AMS256 Homework 3

1. Let $\boldsymbol{A}$ be a $m \times n$ constant matrix and $\boldsymbol{x}$ be a $n \times 1$ random vector. Show that $\text{Cov}(\boldsymbol{A}\boldsymbol{x}) = \boldsymbol{A}\text{Cov}(\boldsymbol{x})\boldsymbol{A}^T$.

2. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be $m \times n$ and $p \times q$ constant matrices, respectively, and $\boldsymbol{x}$ and $\boldsymbol{y}$ be $n \times 1$ and $q \times 1$ random vectors, respectively. Show that $\text{Cov}(\boldsymbol{A}\boldsymbol{x}, \boldsymbol{B}\boldsymbol{y}) = \boldsymbol{A}\text{Cov}(\boldsymbol{x}, \boldsymbol{y})\boldsymbol{B}^T$.

3. Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be $m \times 1$ and $n \times 1$ constant vectors, respectively, and $\boldsymbol{x}$ and $\boldsymbol{y}$ be $m \times 1$ and $n \times 1$ random vectors, respectively. Show that $\text{Cov}(\boldsymbol{x} - \boldsymbol{a}, \boldsymbol{y} - \boldsymbol{b}) = \text{Cov}(\boldsymbol{x}, \boldsymbol{y})$.

4. Let $x_1, \ldots, x_n$ be random variables with $\text{Var}(x_1) = \sigma^2$ and $x_{i+1} = \rho x_i + a$ with a, ρ constants. Find $\text{Cov}(\boldsymbol{x})$ for $\boldsymbol{x} = (x_1, \ldots, x_n)^T$.

5. Let $x_1, \ldots, x_n$ be independent random variables with $\text{E}(x_i) = \mu$ and $\text{Var}(x_i) = \sigma_i^2$. Prove that
$$\frac{\sum_i (x_i - \bar{x})^2}{n(n-1)}$$
is an unbiased estimator of $\text{Var}(\bar{x})$.

6. Let $y_i = \beta x_i + \epsilon_i$, $i = 1, 2$ where $\epsilon_1 \sim \text{N}(0, \sigma^2)$ and $\epsilon_2 \sim \text{N}(0, 2\sigma^2)$, and ϵ_1, ϵ_2 are assumed independent. Let $x_1 = 1$ and $x_2 = -1$. Obtain the weighted least squares estimate of β and its variance.

7. If $x \sim \text{N}(0, \sigma^2)$, write the moment generating function of x and prove that $\mu_3 = 0$ and $\mu_4 = 3\mu_2^2$ where μ_i denotes the $i-$th moment of x.

8. Let $\boldsymbol{y} = (y_1, \ldots, y_n)^T$ such that $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{z} + \boldsymbol{\mu}$ with $\boldsymbol{z} = (z_1, \ldots, z_n)^T$, $z_i \overset{iid}{\sim} \text{N}(0, 1)$ and a $n \times n$ matrix $\boldsymbol{A}$ such that $\Sigma = \boldsymbol{A}\boldsymbol{A}^T$ with Σ positive definite. Show that the density of $\boldsymbol{y}$ is
$$f(\boldsymbol{y}) = (2\pi)^{-n/2}|\Sigma|^{-1/2} \exp\left\{-\frac{1}{2}(\boldsymbol{y} - \boldsymbol{\mu})^T \Sigma^{-1} (\boldsymbol{y} - \boldsymbol{\mu})\right\}.$$

9. Show that if $\boldsymbol{x} \sim \text{N}_k(\boldsymbol{m}, \Sigma)$, then $\text{E}(\boldsymbol{x}) = \boldsymbol{m}$ and $\text{Cov}(\boldsymbol{x}) = \Sigma$.

10. Let $\boldsymbol{y} \sim \text{N}_n(\boldsymbol{\mu}, \Sigma)$ and $\boldsymbol{C}$ a $p \times n$ matrix. Show that $\boldsymbol{C}\boldsymbol{y} \sim \text{N}_p(\boldsymbol{C}\boldsymbol{\mu}, \boldsymbol{C}\Sigma\boldsymbol{C}^T)$.

11. Let $\boldsymbol{y} = (\boldsymbol{y}_1^T, \boldsymbol{y}_2^T)^T$ be an n-dimensional random vector with $\boldsymbol{y} \sim \text{N}(\boldsymbol{m}, \Sigma)$. Assume that $\boldsymbol{y}_1$ and $\boldsymbol{y}_2$ are, respectively, $p-$dimensional and $q-$dimensional vectors with $p + q = n$. In addition, assume that
$$\boldsymbol{m} = \begin{bmatrix} \boldsymbol{m}_1 \\ \boldsymbol{m}_2 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}.$$
Show that $\boldsymbol{y}_1$ and $\boldsymbol{y}_2$ are independent iff $\Sigma_{12} = \Sigma_{21}^T = 0$.

12. Let
$$\begin{bmatrix} X_1 \\ Y_1 \end{bmatrix}, \begin{bmatrix} X_2 \\ Y_2 \end{bmatrix}, \ldots, \begin{bmatrix} X_n \\ Y_n \end{bmatrix}$$
be a random sample from $\text{N}_2(\boldsymbol{\theta}, \Sigma)$. This is, $Z_i \sim \text{N}_2(\boldsymbol{\theta}, \Sigma)$ with $\boldsymbol{Z}_i = (X_i, Y_i)^T$. Assume that $\boldsymbol{Z}_i$ and $\boldsymbol{Z}_j$ are independent for $i \neq j$. Find the joint density of the sample means $\bar{X}$ and $\bar{Y}$.

13. Let $\boldsymbol{Y} \sim \mathrm{N}_n(\theta\mathbf{1}, \Sigma)$, where $\sigma_{i,j} = \sigma^2$ and $\sigma_{i,j} = \sigma^2\rho$ for all i, j, $i \neq j$. Prove the following:

    (a) Σ can be written as $\Sigma = \sigma^2[(\mathbf{1} - \rho)\boldsymbol{I} + \rho\mathbf{1}\mathbf{1}^T]$.

    (b) $\sum_{i=1}^n (Y_i - \bar{Y})^2/[\sigma^2(1-\rho)] \sim \chi^2_{n-1}$.

    (c) $\bar{Y}$ and $\sum_i (Y_i - \bar{Y})^2$ are independent.

14. Let $\boldsymbol{X} = (x_1, x_2, x_3)^T \sim \mathrm{N}(\boldsymbol{\mu}, \Sigma)$ where $\boldsymbol{\mu} = (\mu_1, \mu_2, \mu_3)^T$ and

$$\Sigma = \begin{bmatrix} 1 & \rho & 0 \\ \rho & 1 & \rho \\ 0 & \rho & 1 \end{bmatrix}.$$

    (a) What are the marginal distributions of x_2 and x_3?

    (b) Find the distribution of $(x_1 \mid x_2, x_3)$. Under what condition does this distribution coincide with the marginal distribution of x_1?

    (c) For what value of ρ are the two random variables $x_1+x_2+x_3$ and $x_1-x_2-x_3$ independently distributed?

15. The logarithm of the m.g.f. of a trivariate random vector $\boldsymbol{x} = (x_1, x_2, x_3)^T$ is given by

$$\log M_x(t) = 5t_1^2 + 3t_2^2 + 6t_3^2 - 2t_1t_2 + 4t_1t_3 + 2t_2t_3 + 4t_1 - 2t_2 + t_3.$$

    Show that x has a trivariate normal distribution. Identify the mean and variance-covariance matrix.

16. Let $\boldsymbol{x} = (x_1, x_2, x_3)^T \sim \mathrm{N}(\mathbf{0}, \Sigma)$, where

$$\Sigma = \begin{bmatrix} 4 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix}.$$

    (a) Find the conditional distribution of $(x_1 \mid x_2, x_3)$.

    (b) Find the distribution of $4x_1 - 6x_2 + x_3 - 18$.

17. Suppose that $\boldsymbol{y} \sim \mathrm{N}_3(\boldsymbol{m}, \sigma^2\boldsymbol{I})$. Let

$$\boldsymbol{m} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}, \boldsymbol{A} = \frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix}.$$

    (a) What is the distribution of $\boldsymbol{y}^T\boldsymbol{A}\boldsymbol{y}/\sigma^2$?

    (b) Are $\boldsymbol{y}^T\boldsymbol{A}\boldsymbol{y}$ and $\boldsymbol{B}\boldsymbol{y}$ independent?

    (c) Are $\boldsymbol{y}^T\boldsymbol{A}\boldsymbol{y}$ and $y_1 + y_2 + y_3$ independent?

18. Suppose that $\boldsymbol{y} \sim \mathrm{N}_n(\boldsymbol{m}, \sigma^2\boldsymbol{I})$ and suppose that $\boldsymbol{X}$ is an $n \times p$ matrix of constants with rank $p < n$.

    (a) Show that $\boldsymbol{A} = \boldsymbol{X}(\boldsymbol{X}^T\boldsymbol{X})^{-}\boldsymbol{X}^T$ and $\boldsymbol{I} - \boldsymbol{A}$ are idempotent and find the rank of each.

(b) If $\boldsymbol{m}$ is a linear combination of the columns of $\boldsymbol{X}$, i.e., $\boldsymbol{m} = \boldsymbol{X}\boldsymbol{b}$ for some $\boldsymbol{b}$, find $\mathrm{E}(\boldsymbol{y}^T\boldsymbol{A}\boldsymbol{y})$ and $\mathrm{E}[\boldsymbol{y}^T(\boldsymbol{I} - \boldsymbol{A})\boldsymbol{y}]$.

(c) Find the distributions of $\boldsymbol{y}^T\boldsymbol{A}\boldsymbol{y}/\sigma^2$ and $y^T(\boldsymbol{I} - \boldsymbol{A})\boldsymbol{y}/\sigma^2$.

(d) Show that $\boldsymbol{y}^T\boldsymbol{A}\boldsymbol{y}$ and $\boldsymbol{y}^T(\boldsymbol{I} - \boldsymbol{A})\boldsymbol{y}$ are independent.

(e) Find the distribution of

$$\frac{\boldsymbol{y}^T\boldsymbol{A}\boldsymbol{y}/p}{\boldsymbol{y}^T(\boldsymbol{I} - \boldsymbol{A})\boldsymbol{y}/(n-p)}.$$