AMS256 Homework 4

1. Consider the following two-way ANOVA model:

\[ y_{i,j,k} = \mu + \alpha_i + \beta_j + \epsilon_{i,j,k}. \]

Assume that the response values are given in the following table:

<table>
<thead>
<tr>
<th>Factor B</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Factor A</td>
<td>1</td>
<td>17, 20</td>
<td>15</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>12</td>
<td>11, 14</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>6</td>
<td>17</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>9</td>
<td>4, 6</td>
</tr>
</tbody>
</table>

Note that empty cells mean no observation. Show that \( H_0: \beta_1 = \beta_2 = \beta_3 \) is testable and test \( H_0 \) at the 5% level.

2. Let

\[
\begin{align*}
    y_1 &= \alpha_1 + \epsilon_1, \\
    y_2 &= 2\alpha_2 - \alpha_1 + \epsilon_2, \\
    y_3 &= \alpha_1 + 2\alpha_2 + \epsilon_2,
\end{align*}
\]

where \( \epsilon = (\epsilon_1, \epsilon_2, \epsilon_3)^T \sim N_3(0, \sigma^2 I) \). Derive the F-statistic for testing \( H_0: \alpha_1 = \alpha_2 \).

3. Consider the model

\[
\begin{align*}
    E(y_i) &= \beta_{1,0} + \beta_{1,1}x_i, \quad i = 1, \ldots, n, \\
    E(y_i) &= \beta_{2,0} + \beta_{2,1}x_i, \quad i = n + 1, \ldots, n + m.
\end{align*}
\]

Suppose \( y_i \)'s are independent normal random variables with variance \( \sigma^2 \). Let \( \gamma \) denote the value of \( x \) at which the lines intersect. Find the MLEs of \( \beta_{1,0}, \beta_{1,1}, \beta_{2,0}, \beta_{2,1}, \sigma^2 \) and \( \gamma \). Construct a 95% confidence interval for \( \gamma \). Does such an interval always exist?

4. Consider the balanced one-way ANOVA model. Show that if \( c_i^T \hat{\beta} \) and \( c_j^T \hat{\beta} \) are orthogonal contrasts (i.e., \( c_i^T c_j = 0 \) for \( i \neq j \)), then \( c_i^T \hat{\beta} \) and \( c_j^T \hat{\beta} \) are independent.

5. Consider the one-way ANOVA model

\[ y_{i,j} = \mu + \alpha_i + \epsilon_{i,j} \equiv \mu_i + \epsilon_{i,j}, \]

with \( \epsilon_{i,j} \) i.i.d. \( N(0, 1) \) for \( i = 1 : k \) and \( j = 1 : n_i \). Show that two contrasts \( \hat{\delta} = \sum_{i=1}^k a_i \bar{y}_i \), and \( \hat{\gamma} = \sum_{i=1}^k b_i \bar{y}_i \), are independent if and only if \( \sum_{i=1}^k a_i b_i / n_i = 0 \).

6. Let \( P = X(X^T X)^{-1}X^T \). Verify the following properties of the residual vector \( \hat{\epsilon} \):

(a) \( E(\hat{\epsilon}) = 0 \)
(b) \( \text{Cov}(\hat{\epsilon}) = \sigma^2(I - P) \).
(c) \( \text{Cov}(\hat{\epsilon}, \hat{y}) = \sigma^2(I - P) \).
(d) \( \text{Cov}(\hat{\epsilon}, \hat{\gamma}) = 0 \).