## AMS256 Homework 4

1. Consider the following two-way ANOVA model:

$$
y_{i, j, k}=\mu+\alpha_{i}+\beta_{j}+\epsilon_{i, j, k}
$$

Assume that the response values are given in the following table:

|  |  | Factor B |  |  |
| :---: | :---: | :---: | :---: | :---: |
|  |  | 1 | 2 | 3 |
| Factor | 1 | 17,20 | 15 | 20 |
| A | 2 | 12 |  | 11,14 |
|  | 3 | 6 |  | 17 |
|  | 4 | 9 | 4,6 | 19 |

Note that empty cells mean no observation. Show that $H_{0}: \beta_{1}=\beta_{2}=\beta_{3}$ is testable and test $H_{0}$ at the $5 \%$ level.
2. Let

$$
\begin{aligned}
& y_{1}=\alpha_{1}+\epsilon_{1} \\
& y_{2}=2 \alpha_{2}-\alpha_{2}+\epsilon_{2} \\
& y_{3}=\alpha_{1}+2 \alpha_{2}+\epsilon_{2}
\end{aligned}
$$

where $\boldsymbol{\epsilon}=\left(\epsilon_{1}, \epsilon_{2}, \epsilon_{3}\right)^{T} \sim \mathrm{~N}_{3}\left(0, \sigma^{2} I\right)$. Derive the F-statistic for testing $H_{0}: \alpha_{1}=\alpha_{2}$.
3. Consider the model

$$
\begin{aligned}
& \mathrm{E}\left(y_{i}\right)=\beta_{1,0}+\beta_{1,1} x_{i}, \quad i=1, \ldots, n \\
& \mathrm{E}\left(y_{i}\right)=\beta_{2,0}+\beta_{2,1} x_{i}, \quad i=n+1, \ldots, n+m
\end{aligned}
$$

Suppose $y_{i}$ 's are independent normal random variables with variance $\sigma^{2}$. Let $\gamma$ denote the value of $x$ at which the lines intersect. Find the MLEs of $\beta_{1,0}, \beta_{1,1}, \beta_{2,0}, \beta_{2,1}, \sigma^{2}$ and $\gamma$. Construct a $95 \%$ confidence interval for $\gamma$. Does such an interval always exist?
4. Consider the balanced one-way ANOVA model. Show that if $\boldsymbol{c}_{i}^{T} \hat{\beta}$ and $\boldsymbol{c}_{j}^{T} \hat{\beta}$ are orthogonal contrasts (i.e., $\boldsymbol{c}_{i}^{T} \boldsymbol{c}_{j}=0$ for $i \neq j$ ), then $\boldsymbol{c}_{i}^{T} \hat{\beta}$ and $\boldsymbol{c}_{j}^{T} \hat{\beta}$ are independent.
5. Consider the one-way ANOVA model

$$
y_{i, j}=\mu+\alpha_{i}+\epsilon_{i, j} \equiv \mu_{i}+\epsilon_{i, j}
$$

with $\epsilon_{i, j}$ i.i.d. $\mathrm{N}(0,1)$ for $i=1: k$, and $j=1: n_{i}$. Show that two contrasts $\hat{\delta}=\sum_{i=1}^{k} a_{i} \bar{y}_{i, \cdot}$, and $\hat{\gamma}=\sum_{i=1}^{k} b_{i} \bar{y}_{i, \cdot}$, are independent if and only if $\sum_{i=1}^{k} a_{i} b_{i} / n_{i}=0$.
6. Let $\boldsymbol{P}=\boldsymbol{X}\left(\boldsymbol{X}^{T} \boldsymbol{X}\right)^{-1} \boldsymbol{X}^{T}$. Verify the following properties of the residual vector $\hat{\boldsymbol{\epsilon}}$ :
(a) $\mathrm{E}(\hat{\boldsymbol{\epsilon}})=\mathbf{0}$
(b) $\operatorname{Cov}(\hat{\boldsymbol{\epsilon}})=\sigma^{2}(\boldsymbol{I}-\boldsymbol{P})$.
(c) $\operatorname{Cov}(\hat{\boldsymbol{\epsilon}}, \boldsymbol{y})=\sigma^{2}(\boldsymbol{I}-\boldsymbol{P})$.
(d) $\operatorname{Cov}(\hat{\boldsymbol{\epsilon}}, \hat{\boldsymbol{y}})=\mathbf{0}$.

